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Dressing operator approach to the toroidal model of higher-dimensional dispersionless KP hierarchy

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Abstract
Based upon the dressing operator approach we investigate the twistor theoretical construction, additional symmetries, hodograph solutions, and Miura transformation for the toroidal model of higher-dimensional dispersionless KP hierarchy introduced by Takasaki.

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1. Introduction

Recently, due to active developments in lower-dimensional quantum field theories, the dispersionless Lax hierarchies have attracted much attention from theoretical physicists and mathematicians (see e.g. [1, 9, 17, 18, 20, 34] and references therein). Basically, dispersionless Lax hierarchies can be defined by an algebra of Laurent series (instead of psuedo-differential operators) $\lambda = \sum_i a_i(t)k^i$ with respect to the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial k}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial k}. \tag{1.1}$$

Introducing the projections $\left(\sum_i a_i k^i\right)_{\geqslant l} = \sum_{i\geqslant l} a_i k^i$ and $\left(\sum_i a_i k^i\right)_{<l} = \sum_{i<l} a_i k^i$ then it turns out that, for a Laurent series $L = a_1 k + a_0 + a_{-1}k^{-1} + \cdots$, three well-known classes of dispersionless Lax hierarchies can be constructed as

$$\frac{\partial L}{\partial t_n} = \{(L^n)_{\geqslant l}, L\}, \qquad l = 0, 1, 2 \tag{1.2}$$

where dispersionless Kadomtsev–Petviashvili (dKP) [13, 14], dispersionless modified KP(dmKP) [5, 19] and dispersionless Harry-Dym (dDym) [3, 6, 20] hierarchies correspond to cases $l = 0, 1, 2$, respectively. Besides, the Lax formalism of the dispersionless Toda (dToda) hierarchy [15] was established by considering a pair of Lax operators [29, 31]. In a series of works [29–31, 34] Takasaki and Takebe developed a twistor theoretical method to study the solution structure and symmetries of the dKP and dToda hierarchies; including tau-function, twistor theoretical construction, finite-dimensional reductions, hodograph solutions, additional symmetries and associated w algebras etc. Furthermore, they introduced dressing formulation to the dKP and dToda hierarchies [34], which is a quasi-classical limit of the Sato formalism [26] and is convenient to discuss solutions of finite-dimensional reductions for dispersionless Lax hierarchies [7, 11, 16, 21–23].

In the past few years, there have been a lot of proposals for constructing higher-dimensional dKP hierarchy (see e.g. [4, 8, 10, 12, 24, 28, 32, 33, 35]). These higher-dimensional systems still possess many algebraic structures and deserve to be explored for their underling integrability. Especially, Takasaki proposed a toroidal model of dKP (TdKP) hierarchy [35] for finding a higher-dimensional tau function (or F function in the context of topological field theory). In this toroidal model the extra spatial coordinates are compactified to a two-dimensional torus T^2 and thus the geometry of phase space of the corresponding Poisson bracket is $R^2 \times T^2$. Based on extended Lax formalism, Takasaki investigated the integrability and symmetries of the associated τ function of the toroidal model.

The main purpose of this work is to give a dressing operator approach to Takasaki's TdKP hierarchy. We find that only two classes ($l = 0, 1$) of the aforementioned dispersionless Lax hierarchies can be survived in this higher-dimensional generalization. Motivated by the work [34] for the dKP hierarchy, we shall show that the dressing approach can be generalized to a higher dimension to discuss the solution structure and symmetries of the TdKP hierarchy. Especially, we shall obtain finite-dimensional reductions of the TdKP hierarchy from the twistor construction and explore their hodograph solutions by solving constraint equations for the twistor data.

Our paper is organized as follows. In section 2, we recall the extended Lax formulation of the TdKP proposed by Takasaki. In section 3, we develop the dressing operator approach to the TdKP hierarchy. In section 4, we show that the solution structure of the TdKP hierarchy can be characterized by the twistor theoretical construction. The additional symmetries of the twistor data and associated w algebras are investigated in section 5. In section 6, we present Gelfand–Dickey reductions and their hodograph solutions for the TdKP hierarchy by choosing some suitable twistor data. In section 7, we establish the Miura transformation between the TdKP and TdmKP hierarchies. Section 8 is devoted to the concluding remarks.

2. Extended Lax formulation

The TdKP hierarchy introduced by Takasaki is defined by an algebra of Laurent series $\Lambda = \sum_i a_i(t, x, \theta)k^i$ with respect to the Poisson bracket over the four-dimensional phase space $(k, x, \theta_1, \theta_2)$ [35]

$$\begin{aligned}\{A(k,x,\theta), B(k,x,\theta)\} &= \frac{\partial A}{\partial k}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial k} + \frac{\partial A}{\partial \theta_1}\frac{\partial B}{\partial \theta_2} - \frac{\partial A}{\partial \theta_2}\frac{\partial B}{\partial \theta_1},\\ &\equiv \{A(k,x,\theta), B(k,x,\theta)\}_{kx} + \{A(k,x,\theta), B(k,x,\theta)\}_{\theta}, \qquad (2.1)\end{aligned}$$

where $A, B \in \Lambda$ and $\theta = (\theta_1, \theta_2)$ denotes coordinates on a two-dimensional torus T^2. For a Laurent series $A \in \Lambda$ we denote $\mathrm{ord}(A)$ as the highest order of k of A. Then it is easy to see that $\mathrm{ord}(\{A, B\}) = \mathrm{ord}(A) + \mathrm{ord}(B)$. This is different from the original Poisson bracket (without the extra coordinates θ_1 and θ_2) where $\mathrm{ord}(\{A, B\}) = \mathrm{ord}(A) + \mathrm{ord}(B) - 1$. From Lie–Poisson algebra point of view [27], the decomposition $\Lambda = \Lambda_{\geqslant l} \oplus \Lambda_{<l}$ with respect to

the Poisson bracket (2.1) is a Lie subalgebra decomposition only for $l = 0, 1$. Namely

$$\{\Lambda_{\geqslant l}, \Lambda_{\geqslant l}\} \subset \Lambda_{\geqslant l}, \qquad \{\Lambda_{<l}, \Lambda_{<l}\} \subset \Lambda_{<l}, \qquad l = 0, 1.$$

The extended Lax representation of the TdKP hierarchy ($l = 0$) is given by [35]

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_{n\alpha}} &= \{\mathcal{B}_{n\alpha}, \mathcal{L}\}, & \frac{\partial \mathcal{M}}{\partial t_{n\alpha}} &= \{\mathcal{B}_{n\alpha}, \mathcal{M}\}, \\ \frac{\partial \mathcal{U}}{\partial t_{n\alpha}} &= \{\mathcal{B}_{n\alpha}, \mathcal{U}\}, & \frac{\partial \mathcal{V}}{\partial t_{n\alpha}} &= \{\mathcal{B}_{n\alpha}, \mathcal{V}\}, \end{aligned} \tag{2.2}$$

with constraints

$$\{\mathcal{L}, \mathcal{M}\} = \{\mathcal{U}, \mathcal{V}\} = 1, \qquad \{\mathcal{L}, \mathcal{U}\} = \{\mathcal{L}, \mathcal{V}\} = \{\mathcal{M}, \mathcal{U}\} = \{\mathcal{M}, \mathcal{V}\} = 0, \tag{2.3}$$

where the time variables $t_{n\alpha}$ have a double index (n, α) with $n = 0, 1, 2, \ldots, \alpha = 0, \pm 1, \pm 2, \ldots, \mathcal{B}_{n\alpha} \equiv (\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\geqslant 0}$. The four fundamental Laurent series $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ have the form

$$\begin{aligned} \mathcal{L} &= k + \sum_{n=1}^{\infty} g_{n+1}(t, x, \theta) k^{-n}, \\ \mathcal{M} &= \sum_{n,\alpha} n t_{n\alpha} \mathcal{L}^{n-1} \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}} + x + \sum_{n=1}^{\infty} h_n(t, x, \theta) \mathcal{L}^{-n-1}, \\ \mathcal{U} &= \theta_1 + \sum_{n=1}^{\infty} u_n(t, x, \theta) \mathcal{L}^{-n}, \\ \mathcal{V} &= \sum_{n,\alpha} \mathrm{i}\alpha t_{n\alpha} \mathcal{L}^{n} \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}} + \theta_2 + \sum_{n=1}^{\infty} v_n(t, x, \theta) \mathcal{L}^{-n}. \end{aligned} \tag{2.4}$$

Since $\mathcal{B}_{10} = k$ and $\mathcal{B}_{0\alpha} = \mathrm{e}^{\mathrm{i}\alpha\theta_1}$, we have $\partial\mathcal{L}/\partial t_{10} = \partial\mathcal{L}/\partial x$ and $\partial\mathcal{L}/\partial t_{0\alpha} = \mathrm{i}\alpha \,\mathrm{e}^{\mathrm{i}\alpha\theta_1} \partial\mathcal{L}/\partial\theta_2$ etc. This leads to the linear combinations $t_{10} + x$ and $\theta_2 + \sum_\alpha \mathrm{i}\alpha t_{0\alpha} \,\mathrm{e}^{\mathrm{i}\alpha\theta_1}$ in all quantities of the hierarchy. We remark that the coefficient functions h_n, g_n, u_n, v_n are assumed to be double periodic functions on T^2 and that is the reason why the model is named toroidal model.

It would be useful to introduce the notion of gradings for the coefficient functions (g_n, h_n, u_n, v_n). If we set the gradings of $(k, x, \theta_1, \theta_2)$ as $[k] = 1, [x] = -1$, and $[\theta_1] = [\theta_2] = 0$ then due to the homogeneity of gradings for the Laurent series $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ we have $[g_n] = [h_n] = [u_n] = [v_n] = n$. We also use the following conventions throughout the paper.

$$\partial_{n\alpha} f = \frac{\partial f}{\partial t_{n\alpha}}, \qquad \mathrm{res}\left(\sum_i a_i k^i\right) = a_{-1}, \qquad \left(\sum_i a_i k^i\right)_{[j]} = a_j.$$

Proposition 1 ([35]). *The Lax equations for $\mathcal{L}$ and $\mathcal{U}$ are equivalent to the zero curvature equations*

$$\frac{\partial \mathcal{B}_{m\beta}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{B}_{n\alpha}}{\partial t_{m\beta}} + \{\mathcal{B}_{m\beta}, \mathcal{B}_{n\alpha}\} = 0, \tag{2.5}$$

or

$$\frac{\partial \mathcal{B}^-_{m\beta}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{B}^-_{n\alpha}}{\partial t_{m\beta}} - \{\mathcal{B}^-_{m\beta}, \mathcal{B}^-_{n\alpha}\} = 0,$$

where $\mathcal{B}^-_{n\alpha} \equiv (\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}$.

The first nontrivial equations ($n = 2, m = 1, \alpha = 1, \beta = 1$) and ($n = 3, m = 2, \alpha = 0, \beta = 0$) are given by

$$
\begin{aligned}
(2-\mathrm{i}\theta_1)\partial_{11}g_2 &= \mathrm{i}\int^x \partial_{21}\partial_{\theta_2}g_2 + \mathrm{e}^{\mathrm{i}\theta_1}\Bigg((1-\mathrm{i}\theta_1)(2-\mathrm{i}\theta_1)g_{2x} + (1-\mathrm{i}\theta_1)g_2\int^x \partial^2_{\theta_2}g_2 \\
&\quad - \partial_{\theta_2}g_2\int^x \partial_{\theta_2}g_2 - \frac{1}{2}\left(\int^x \partial_{\theta_2}g_2\right)^2\left(\int^x \partial^2_{\theta_2}g_2\right)\Bigg), \\
\frac{3}{4}\partial^2_{20}g_2 &= \left(\partial_{30}g_2 - 3g_2g_{2x} + \frac{3}{2}\left\{g_2, \int^x \partial_{20}g_2\right\}_\theta\right)_x.
\end{aligned}
\tag{2.6}
$$

In particular, the second equation of (2.6) can be regarded as a higher-dimensional generalization of the ordinary (2+1)-dKP equation, $U_t = 3UU_x + \frac{3}{4}\partial_x^{-1}U_{yy}$, if one sets $t = t_{30}$ and $y = t_{20}$.

For arbitrary Laurent series $X, Y \in \Lambda$, denoting $\mathrm{ad}X(Y) = \{X, Y\}$, one can verify the following useful identities.

Lemma 2 ([34]).

(a) For all $X, Y \in \Lambda$

$$\partial_{n\alpha}\,\mathrm{e}^{\mathrm{ad}Y}(X) = \mathrm{e}^{\mathrm{ad}Y}(\partial_{n\alpha}X) + \{\nabla_{t_{n\alpha},Y}Y, \mathrm{e}^{\mathrm{ad}Y}(X)\},$$

where

$$\nabla_{t_{n\alpha},Z}W \equiv \sum_{k=0}^{\infty}\frac{(\mathrm{ad}Z)^k}{(k+1)!}\partial_{n\alpha}W = \frac{\mathrm{e}^{\mathrm{ad}Z}-1}{\mathrm{ad}Z}\partial_{n\alpha}W.$$

(b) For all $X, Y \in \Lambda$

$$\nabla_{t_{n\alpha},H(X,Y)}H(X,Y) = \nabla_{t_{n\alpha},X}X + \mathrm{e}^{\mathrm{ad}X}(\nabla_{t_{n\alpha},Y}Y),$$

where $H(X, Y)$ *is the Hausdorff series defined by* $\mathrm{e}^{\mathrm{ad}H(X,Y)} = \mathrm{e}^{\mathrm{ad}X}\,\mathrm{e}^{\mathrm{ad}Y}$.

(c) For $X, Y, Z \in \Lambda$ *define*

$$\widetilde{X} = \mathrm{e}^{\mathrm{ad}Z}(X) + \nabla_{t_{m\beta},Z}Z, \qquad \widetilde{Y} = \mathrm{e}^{\mathrm{ad}Z}(Y) + \nabla_{t_{n\alpha},Z}Z,$$

then

$$\partial_{n\alpha}X - \partial_{m\beta}Y + \{X, Y\} = 0 \quad\Leftrightarrow\quad \partial_{n\alpha}\widetilde{X} - \partial_{m\beta}\widetilde{Y} + \{\widetilde{X}, \widetilde{Y}\} = 0.$$

3. Dressing operator approach

In this section we like to show that just as the dKP hierarchy, the TdKP hierarchy can be formulated in a dressing form so that the dynamics of $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ can be encoded by a dressing function.

Proposition 3. *Let* $\mathcal{L}$ *and* $\mathcal{U}$ *satisfy the Lax equations (2.2) and the canonical relation* $\{\mathcal{L}, \mathcal{U}\} = 0$. *Then there exists a Lourant series of the form*

$$\varphi = \varphi_1(t,\theta)k^{-1} + \varphi_2(t,\theta)k^{-2} + \cdots \tag{3.1}$$

such that

$$\mathcal{L} = \mathrm{e}^{\mathrm{ad}\varphi}(k), \qquad \mathcal{U} = \mathrm{e}^{\mathrm{ad}\varphi}(\theta_1), \tag{3.2}$$

where φ *satisfies the equation*

$$\nabla_{t_{n\alpha},\varphi}\varphi = -(\mathrm{e}^{\mathrm{ad}\varphi}(k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}))_{\leqslant -1}. \tag{3.3}$$

Conversely, if $\varphi = \sum_{n=1} \varphi_n(t,\theta)k^{-n}$ *satisfies equation (3.3) then* $\mathcal{L}$ *and* $\mathcal{U}$ *defined in (3.2) satisfy the Lax equation (2.2).*

Proof. Given a Laurent series of the form $\phi = \sum_{n=1} \phi_n(t,\theta)k^{-n}$ we can define two Laurent series $\mathcal{L} = \mathrm{e}^{\mathrm{ad}\phi}(k)$ and $\mathcal{U} = \mathrm{e}^{\mathrm{ad}\phi}(\theta_1)$ which have the form defined in (2.4) and satisfy the canonical relation $\{\mathcal{L},\mathcal{U}\} = 0$. If $\mathcal{L}$ and $\mathcal{U}$ satisfy the Lax equations (2.2) then by lemma 2(a)

$$\partial_{n\alpha}\,\mathrm{e}^{\mathrm{ad}\phi}(k) = \left\{\nabla_{t_{n\alpha},\phi}\phi, \mathcal{L}\right\} = \{\mathcal{B}_{n\alpha}, \mathcal{L}\},$$

which implies

$$\left\{\mathrm{e}^{-\mathrm{ad}\phi}\left(\mathcal{B}_{n\alpha} - \nabla_{t_{n\alpha},\phi}\phi\right), k\right\} = 0.$$

Similarly, for $\mathcal{U}$, we have

$$\left\{\mathrm{e}^{-\mathrm{ad}\phi}\left(\mathcal{B}_{n\alpha} - \nabla_{t_{n\alpha},\phi}\phi\right), \theta_1\right\} = 0.$$

That means

$$\mathcal{B}^{\phi}_{n\alpha} \equiv \mathrm{e}^{-\mathrm{ad}\phi}\left(\mathcal{B}_{n\alpha} - \nabla_{t_{n\alpha},\phi}\phi\right) = k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{m=1} b_{n\alpha,m}k^{-m}$$

does not depend on x and θ_2, namely $\partial b_{n\alpha,m}/\partial x = \partial b_{n\alpha,m}/\partial\theta_2 = 0$. On the other hand, by lemma 2(c), $\mathcal{B}^{\phi}_{n\alpha}$ satisfies the zero curvature condition (2.5) and thus $\partial_{n\alpha}\mathcal{B}^{\phi}_{m\beta} = \partial_{m\beta}\mathcal{B}^{\phi}_{n\alpha}$. This implies $\mathcal{B}^{\phi}_{n\alpha} = k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \partial_{n\alpha}\phi'$ where ϕ' is a Laurent series of the form (3.1) with coefficients ϕ'_i do not depend on x and θ_2. Let $\mathrm{e}^{\mathrm{ad}\varphi} = \mathrm{e}^{\mathrm{ad}\phi}\,\mathrm{e}^{\mathrm{ad}\phi'}$, then by lemma 2(b)

$$\begin{aligned}
\mathrm{e}^{-\mathrm{ad}\varphi}\left(\mathcal{B}_{n\alpha} - \nabla_{t_{n\alpha},\varphi}\varphi\right) &= \mathrm{e}^{-\mathrm{ad}\phi'}\,\mathrm{e}^{-\mathrm{ad}\phi}\left(\mathcal{B}_{n\alpha} - \nabla_{t_{n\alpha}\phi}\phi - \mathrm{e}^{\mathrm{ad}\phi}\nabla_{t_{n\alpha},\phi'}\phi'\right),\\
&= \mathrm{e}^{-\mathrm{ad}\phi'}\left(\mathcal{B}^{\phi}_{n\alpha} - \partial_{n\alpha}\phi'\right),\\
&= k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}.
\end{aligned}$$

Conversely, it is easy to show that the dressing form (3.2) with φ defined by (3.3) satisfies the Lax equations (2.2). □

Remark 1. Note that the only ambiguity of the dressing function φ coming from that of ϕ', which can be absorbed into the transformation $\mathrm{e}^{\mathrm{ad}\varphi} \mapsto \mathrm{e}^{\mathrm{ad}\varphi}\,\mathrm{e}^{\mathrm{ad}\psi} = \mathrm{e}^{\mathrm{ad}H(\varphi,\psi)}$ with a suitable Laurent series $\psi(k,\theta_1) = \sum_{j=1}\psi_j(\theta_1)k^{-j}$.

Having constructed the dressing form for the Laurent series $\mathcal{L}$ and $\mathcal{U}$, we can introduce the associated Orlov operators $\mathcal{M}$ and $\mathcal{V}$ defined in (2.4) as follows

$$\begin{aligned}
\mathcal{M} &= \mathrm{e}^{\mathrm{ad}\varphi}\,\mathrm{e}^{t(k,\theta_1)}(x) = \mathrm{e}^{\mathrm{ad}\varphi}\left(x + \sum_{n,\alpha} n t_{n\alpha}k^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\right),\\
\mathcal{V} &= \mathrm{e}^{\mathrm{ad}\varphi}\,\mathrm{e}^{t(k,\theta_1)}(\theta_2) = \mathrm{e}^{\mathrm{ad}\varphi}\left(\theta_2 + \sum_{n,\alpha}\mathrm{i}\alpha t_{n\alpha}k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\right),
\end{aligned} \tag{3.4}$$

where the dressing function φ is defined by (3.3) and $t(k,\theta_1) = \sum_{n\alpha} t_{n\alpha}k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}$.

Proposition 4. *The operators* $\mathcal{M}$ *and* $\mathcal{V}$ *defined above satisfy the Lax equations*

$$\frac{\partial\mathcal{M}}{\partial t_{n\alpha}} = \{\mathcal{B}_{n\alpha}, \mathcal{M}\}, \qquad \frac{\partial\mathcal{V}}{\partial t_{n\alpha}} = \{\mathcal{B}_{n\alpha}, \mathcal{V}\},$$

and the canonical relations (2.3). Conversely, if the series $\mathcal{M}$ *and* $\mathcal{V}$ *of the form (2.4) satisfy the Lax equations (2.2) and the canonical relations (2.3), then there exists a Laurent series of the form* $\varphi = \sum_{n=1}\varphi_n k^{-n}$ *such that* $\mathcal{M}$ *and* $\mathcal{V}$ *can be expressed in dressing form (3.4).*

Proof. If $\mathcal{M}$ and $\mathcal{V}$ are defined by the dressing forms (3.4), then the canonical Poisson relation (2.3) can easily be checked since

$$\{\mathcal{L},\mathcal{M}\} = \mathrm{e}^{\mathrm{ad}\varphi}\left\{k, x+\sum_n n t_{n\alpha} k^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\right\} = 1,$$

$$\{\mathcal{U},\mathcal{V}\} = \mathrm{e}^{\mathrm{ad}\varphi}\left\{\theta_1, \theta_2+\sum_n \mathrm{i}\alpha t_{n\alpha} k^{n}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\right\} = 1,$$

and $\{\mathcal{L},\mathcal{U}\} = 0$, etc. On the other hand, using lemma 2(a), the canonical Poisson relation (2.3), (3.2) and (3.3) we have

$$\partial_{n\alpha}\mathcal{M} = n\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}} + \left\{\nabla_{t_{n\alpha},\varphi}\varphi, \mathcal{M}\right\} = \{\mathcal{B}_{n\alpha},\mathcal{M}\}.$$

The proof for $\mathcal{V}$ is similar. Conversely, let φ^0 be the dressing function defined in (3.2). From the dressing form of $\mathcal{L}$ and $\mathcal{U}$ and the canonical Poisson relation $\{\mathcal{L},\mathcal{M}\} = 1$, we have

$$1 = \mathrm{e}^{-\mathrm{ad}\varphi^0}\{\mathcal{L},\mathcal{M}\} = \left\{k, \mathrm{e}^{-\mathrm{ad}\varphi^0}(\mathcal{M})\right\}.$$

Thus $\mathrm{e}^{-\mathrm{ad}\varphi^0}(\mathcal{M})$ has the form

$$\mathrm{e}^{-\mathrm{ad}\varphi^0}(\mathcal{M}) = \sum_{n\alpha} n t_{n\alpha} k^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + x + \sum_{j=1} q_j k^{-j-1},$$

where q_j do not depend on x, i.e. $\partial q_j/\partial x = 0$. Similarly, from $\{\mathcal{U},\mathcal{M}\} = 0$ we have $\partial q_j/\partial\theta_2 = 0$. Moreover,

$$\begin{aligned}\partial_{n\alpha}\,\mathrm{e}^{-\mathrm{ad}\varphi^0}(\mathcal{M}) &= \mathrm{e}^{-\mathrm{ad}\varphi^0}\left(\partial_{n\alpha}\mathcal{M} - \left\{\nabla_{t_{n\alpha}\varphi^0}\varphi^0, \mathcal{M}\right\}\right),\\ &= \left\{k^n\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}, \mathrm{e}^{-\mathrm{ad}\varphi^0}(\mathcal{M})\right\},\\ &= n k^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1},\end{aligned}$$

which implies $\partial q_j/\partial t_{n\alpha} = 0$ and thus $q_j = q_j(\theta_1)$. Therefore

$$\mathcal{M} = \mathrm{e}^{\mathrm{ad}\varphi^0}\,\mathrm{e}^{\mathrm{ad}\psi}\left(\sum_{n\alpha} n t_{n\alpha} k^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + x\right) = \mathrm{e}^{\mathrm{ad}\varphi^0}\,\mathrm{e}^{\mathrm{ad}t(k,\theta_1)}\,\mathrm{e}^{\mathrm{ad}\psi}(x),$$

where $\psi(k,\theta_1)$ is a Laurent series defined by $\psi = \sum_{j=1}\psi_j(\theta_1)k^{-j}$ with $\psi_j = -q_j/j$. On the other hand, from $\{\mathcal{U},\mathcal{V}\} = 1$ and $\{\mathcal{L},\mathcal{V}\} = 0$ we have

$$\mathcal{V} = \mathrm{e}^{\mathrm{ad}\varphi^0}\,\mathrm{e}^{\mathrm{ad}\psi'}\left(\sum_{n\alpha} \mathrm{i}\alpha t_{n\alpha} k^{n}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \theta_2\right) = \mathrm{e}^{\mathrm{ad}\varphi^0}\,\mathrm{e}^{\mathrm{ad}t(k,\theta_1)}\,\mathrm{e}^{\mathrm{ad}\psi'}(\theta_2),$$

where $\psi'(k,\theta_1)$ is a Laurent series of the form $\psi' = \sum_{j=1}\psi'_j(\theta_1)k^{-j}$. Now the remaining task is to show that the dressing functions for $\mathcal{M}$ and $\mathcal{V}$ are equivalent up to a trivial gauge transformation, namely

$$\mathrm{e}^{\mathrm{ad}\psi} = \mathrm{e}^{\mathrm{ad}\psi'}\,\mathrm{e}^{\mathrm{ad}\eta},$$

where $\eta = \sum_{j=1}\eta_j k^{-j}$ with $\eta_j =$ constant. To see this, from $\{\mathcal{M},\mathcal{V}\} = 0$ we have

$$0 = \mathrm{e}^{-\mathrm{ad}\psi'}\,\mathrm{e}^{-\mathrm{ad}t(k,\theta_1)}\,\mathrm{e}^{-\mathrm{ad}\varphi^0}\{\mathcal{M},\mathcal{V}\} = \{\mathrm{e}^{-\mathrm{ad}\psi'}\,\mathrm{e}^{\mathrm{ad}\psi}(x), \theta_2\}.$$

Let $\mathrm{e}^{-\mathrm{ad}\psi'}\,\mathrm{e}^{\mathrm{ad}\psi} = \mathrm{e}^{\mathrm{ad}\eta}$ where $\eta(k,\theta_1) = H(-\psi',\psi) = \sum_{j=1}\eta_j(\theta_1)k^{-j}$. Then the above equation shows that

$$0 = \frac{\partial}{\partial\theta_1}\,\mathrm{e}^{\mathrm{ad}\eta}(x) = \left\{\frac{\partial\eta}{\partial\theta_1}, x\right\}.$$

Hence $\partial\eta_j/\partial\theta_1 = 0$ (i.e. $\eta_j = $ constant) and $\varphi = H\big(\varphi^0, -\sum_{j=1} q_j k^{-j}/j\big)$ gives the desired dressing function. □

Introducing the dressing function φ enables us to express the coefficients φ_j in terms of g_j and h_j. Observing that

$$\begin{aligned}
e^{\mathrm{ad}\varphi}(x) &= x + \sum_{n=1} h_n \mathcal{L}^{-n-1}, \\
&= x + \sum_{n=1} (h_n + (\text{polynomials in } \{g_2, \ldots, g_{n-1}, h_1, \ldots, h_{n-2}\}))k^{-n-1}, \\
&= x + \sum_{n=1} (-n\varphi_n + (\text{differential polynomials of } \{\varphi_1, \ldots, \varphi_{n-1}\}))k^{-n-1},
\end{aligned}$$

where the differential is taken with respect to x and θ. By induction we have

$$\varphi_n = -\frac{h_n}{n} + (\text{differential polynomials of } \{g_2, \ldots, g_{n-1}, h_1, \ldots, h_{n-2}\}).$$

4. Twistor construction

In the following, we shall show that the solution structure of the TdKP hierarchy can be characterized by the Riemann–Hilbert problem.

Proposition 5. *Given a set of functions* $(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})$ *on the phase space* $(k, x, \theta_1, \theta_2)$ *which satisfy the Poisson relations*

$$\begin{aligned}
&\{f^{(1)}, f^{(2)}\} = \{f^{(3)}, f^{(4)}\} = 1, \\
&\{f^{(i)}, f^{(j)}\} = 0, \qquad \text{otherwise.}
\end{aligned} \tag{4.1}$$

If $\mathcal{L}, \mathcal{M}, \mathcal{U}$, *and* $\mathcal{V}$ *have Laurent series of the form (2.4) then the following conditions*

$$f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = 0, \qquad i = 1, 2, 3, 4 \tag{4.2}$$

give a solution $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ *of the TdKP hierarchy. We call* $(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})$ *the twistor data of the corresponding solution.*

In view of (2.3) one can think of the twistor data as a canonical transformation $(k, x, \theta_1, \theta_2) \to (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})$.

Proof. Let $f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V}) = \hat{f}^{(i)}(k, x, \theta)$ which contains only non-negative powers of k. Differentiating $f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ with respect to k, x, θ_1 and θ_2, respectively we obtain

$$\begin{pmatrix}
\frac{\partial f^{(1)}}{\partial \mathcal{L}} & \frac{\partial f^{(1)}}{\partial \mathcal{M}} & \frac{\partial f^{(1)}}{\partial \mathcal{U}} & \frac{\partial f^{(1)}}{\partial \mathcal{V}} \\
\frac{\partial f^{(2)}}{\partial \mathcal{L}} & \frac{\partial f^{(2)}}{\partial \mathcal{M}} & \frac{\partial f^{(2)}}{\partial \mathcal{U}} & \frac{\partial f^{(2)}}{\partial \mathcal{V}} \\
\frac{\partial f^{(3)}}{\partial \mathcal{L}} & \frac{\partial f^{(3)}}{\partial \mathcal{M}} & \frac{\partial f^{(3)}}{\partial \mathcal{U}} & \frac{\partial f^{(3)}}{\partial \mathcal{V}} \\
\frac{\partial f^{(4)}}{\partial \mathcal{L}} & \frac{\partial f^{(4)}}{\partial \mathcal{M}} & \frac{\partial f^{(4)}}{\partial \mathcal{U}} & \frac{\partial f^{(4)}}{\partial \mathcal{V}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial k} & \frac{\partial \mathcal{L}}{\partial x} & \frac{\partial \mathcal{L}}{\partial \theta_1} & \frac{\partial \mathcal{L}}{\partial \theta_2} \\
\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{M}}{\partial x} & \frac{\partial \mathcal{M}}{\partial \theta_1} & \frac{\partial \mathcal{M}}{\partial \theta_2} \\
\frac{\partial \mathcal{U}}{\partial k} & \frac{\partial \mathcal{U}}{\partial x} & \frac{\partial \mathcal{U}}{\partial \theta_1} & \frac{\partial \mathcal{U}}{\partial \theta_2} \\
\frac{\partial \mathcal{V}}{\partial k} & \frac{\partial \mathcal{V}}{\partial x} & \frac{\partial \mathcal{V}}{\partial \theta_1} & \frac{\partial \mathcal{V}}{\partial \theta_2}
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial \hat{f}^{(1)}}{\partial k} & \frac{\partial \hat{f}^{(1)}}{\partial x} & \frac{\partial \hat{f}^{(1)}}{\partial \theta_1} & \frac{\partial \hat{f}^{(1)}}{\partial \theta_2} \\
\frac{\partial \hat{f}^{(2)}}{\partial k} & \frac{\partial \hat{f}^{(2)}}{\partial x} & \frac{\partial \hat{f}^{(2)}}{\partial \theta_1} & \frac{\partial \hat{f}^{(2)}}{\partial \theta_2} \\
\frac{\partial \hat{f}^{(3)}}{\partial k} & \frac{\partial \hat{f}^{(3)}}{\partial x} & \frac{\partial \hat{f}^{(3)}}{\partial \theta_1} & \frac{\partial \hat{f}^{(3)}}{\partial \theta_2} \\
\frac{\partial \hat{f}^{(4)}}{\partial k} & \frac{\partial \hat{f}^{(4)}}{\partial x} & \frac{\partial \hat{f}^{(4)}}{\partial \theta_1} & \frac{\partial \hat{f}^{(4)}}{\partial \theta_2}
\end{pmatrix} \tag{4.3}$$

or $\mathbf{AB} = \mathbf{C}$ for short. Due to the Poisson relations (4.1) we know that $\mathbf{A}$ is a symplectic matrix, that is $\mathbf{A}^t\mathbf{J}\mathbf{A} = \mathbf{J}$ where $\mathbf{J}$ is the canonical symplectic matrix defined by

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{J}^2 = -\mathbf{I}$$

and $\mathbf{A}^t$ denotes the transpose of the matrix $\mathbf{A}$. Using (4.3) and its transpose we have

$$\mathbf{B}^t\mathbf{J}\mathbf{B} = \mathbf{C}^t\mathbf{J}\mathbf{C},$$

where the matrix elements $(\mathbf{C}^t\mathbf{J}\mathbf{C})_{ij}$ contain only non-negative powers of k while $(\mathbf{B}^t\mathbf{J}\mathbf{B})_{ij}$ negative powers of k except

$$(\mathbf{B}^t\mathbf{J}\mathbf{B})_{12} = 1 + (\text{negative powers in } k), \qquad (\mathbf{B}^t\mathbf{J}\mathbf{B})_{34} = 1 + (\text{negative powers in } k).$$

Hence $\mathbf{B}$ is also a symplectic matrix, i.e. $\mathbf{B}^t\mathbf{J}\mathbf{B} = \mathbf{J}$. This completes the proof for the Poisson relations (2.3). To prove the Lax equations (2.2), differentiating $f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ with respect to $t_{n\alpha}$ we obtain

$$\mathbf{A}\begin{pmatrix} \frac{\partial\mathcal{L}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{M}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{U}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{V}}{\partial t_{n\alpha}} \end{pmatrix} = \begin{pmatrix} \frac{\partial\hat{f}^{(1)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(2)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(3)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(4)}}{\partial t_{n\alpha}} \end{pmatrix},$$

which together with $\mathbf{AB} = \mathbf{C}$ implies

$$\mathbf{B}^{-1}\begin{pmatrix} \frac{\partial\mathcal{L}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{M}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{U}}{\partial t_{n\alpha}} \\ \frac{\partial\mathcal{V}}{\partial t_{n\alpha}} \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} \frac{\partial\hat{f}^{(1)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(2)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(3)}}{\partial t_{n\alpha}} \\ \frac{\partial\hat{f}^{(4)}}{\partial t_{n\alpha}} \end{pmatrix}, \tag{4.4}$$

where

$$\mathbf{B}^{-1} = -\mathbf{J}\mathbf{B}^t\mathbf{J} = \begin{pmatrix} \frac{\partial\mathcal{M}}{\partial x} & -\frac{\partial\mathcal{L}}{\partial x} & \frac{\partial\mathcal{V}}{\partial x} & -\frac{\partial\mathcal{U}}{\partial x} \\ -\frac{\partial\mathcal{M}}{\partial k} & \frac{\partial\mathcal{L}}{\partial k} & -\frac{\partial\mathcal{V}}{\partial k} & \frac{\partial\mathcal{U}}{\partial k} \\ \frac{\partial\mathcal{M}}{\partial\theta_2} & -\frac{\partial\mathcal{L}}{\partial\theta_2} & \frac{\partial\mathcal{V}}{\partial\theta_2} & -\frac{\partial\mathcal{U}}{\partial\theta_2} \\ -\frac{\partial\mathcal{M}}{\partial\theta_1} & \frac{\partial\mathcal{L}}{\partial\theta_1} & -\frac{\partial\mathcal{V}}{\partial\theta_1} & \frac{\partial\mathcal{U}}{\partial\theta_1} \end{pmatrix}$$

and $\mathbf{C}^{-1}$ can be constructed from $\mathbf{B}^{-1}$ just by replacing $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ by $(\hat{f}^{(1)}, \hat{f}^{(2)}, \hat{f}^{(3)}, \hat{f}^{(4)})$. It is easy to see that each entry on the right-hand side of (4.4) contains only non-negative powers of k. For the first entry on the left-hand side of (4.4), we have

$$\begin{aligned}
(\text{LHS})_1 &= \frac{\partial\mathcal{M}}{\partial x}\frac{\partial\mathcal{L}}{\partial t_{n\alpha}} - \frac{\partial\mathcal{L}}{\partial x}\frac{\partial\mathcal{M}}{\partial t_{n\alpha}} + \frac{\partial\mathcal{V}}{\partial x}\frac{\partial\mathcal{U}}{\partial t_{n\alpha}} - \frac{\partial\mathcal{U}}{\partial x}\frac{\partial\mathcal{V}}{\partial t_{n\alpha}}, \\
&= \frac{\partial\mathcal{L}}{\partial t_{n\alpha}}\left(\frac{\partial\mathcal{M}}{\partial\mathcal{L}}\bigg|_{\mathcal{U}}\frac{\partial\mathcal{L}}{\partial x} + \sum_{m\beta} \mathrm{i}\beta m t_{m\beta}\mathcal{L}^{m-1}\,\mathrm{e}^{\mathrm{i}\beta\mathcal{U}}\frac{\partial\mathcal{U}}{\partial x} + \sum_j \frac{\partial h_j}{\partial x}\mathcal{L}^{-j-1}\right) \\
&\quad - \frac{\partial\mathcal{L}}{\partial x}\left(\frac{\partial\mathcal{M}}{\partial\mathcal{L}}\bigg|_{\mathcal{U}}\frac{\partial\mathcal{L}}{\partial t_{n\alpha}} + n\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}} + \sum_{m\beta} \mathrm{i}\beta m t_{m\beta}\mathcal{L}^{m-1}\,\mathrm{e}^{\mathrm{i}\beta\mathcal{U}}\frac{\partial\mathcal{U}}{\partial t_{n\alpha}} + \sum_j \frac{\partial h_j}{\partial t_{n\alpha}}\mathcal{L}^{-j-1}\right) \\
&\quad + \frac{\partial\mathcal{U}}{\partial t_{n\alpha}}\left(\frac{\partial\mathcal{V}}{\partial\mathcal{L}}\bigg|_{\mathcal{U}}\frac{\partial\mathcal{L}}{\partial x} - \sum_{m\beta} \beta^2 t_{m\beta}\mathcal{L}^m\,\mathrm{e}^{\mathrm{i}\beta\mathcal{U}}\frac{\partial\mathcal{U}}{\partial x} + \sum_j \frac{\partial v_j}{\partial x}\mathcal{L}^{-j}\right)
\end{aligned}$$

$$- \frac{\partial \mathcal{U}}{\partial x}\left(\frac{\partial \mathcal{V}}{\partial \mathcal{L}}\bigg|_{\mathcal{U}} \frac{\partial \mathcal{L}}{\partial t_{n\alpha}} + \mathrm{i}\alpha \mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha \mathcal{U}} - \sum_{m\beta} \beta^2 t_{m\beta} \mathcal{L}^m \,\mathrm{e}^{\mathrm{i}\beta \mathcal{U}} \frac{\partial \mathcal{U}}{\partial t_{n\alpha}} + \sum_j \frac{\partial v_j}{\partial t_{n\alpha}} \mathcal{L}^{-j} \right),$$

$$= -\frac{\partial \mathcal{B}_{n\alpha}}{\partial x} + (\text{negative powers in } k) = (\text{RHS})_1.$$

Hence

$$(\text{LHS})_1 = \frac{\partial \mathcal{M}}{\partial x}\frac{\partial \mathcal{L}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{V}}{\partial x}\frac{\partial \mathcal{U}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{U}}{\partial x}\frac{\partial \mathcal{V}}{\partial t_{n\alpha}} = -\frac{\partial \mathcal{B}_{n\alpha}}{\partial x}.$$

Similarly, we have

$$(\text{LHS})_2 = -\frac{\partial \mathcal{M}}{\partial k}\frac{\partial \mathcal{L}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{V}}{\partial k}\frac{\partial \mathcal{U}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{U}}{\partial k}\frac{\partial \mathcal{V}}{\partial t_{n\alpha}} = \frac{\partial \mathcal{B}_{n\alpha}}{\partial k},$$

$$(\text{LHS})_3 = \frac{\partial \mathcal{M}}{\partial \theta_2}\frac{\partial \mathcal{L}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{L}}{\partial \theta_2}\frac{\partial \mathcal{M}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{V}}{\partial \theta_2}\frac{\partial \mathcal{U}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{U}}{\partial \theta_2}\frac{\partial \mathcal{V}}{\partial t_{n\alpha}} = -\frac{\partial \mathcal{B}_{n\alpha}}{\partial \theta_2},$$

$$(\text{LHS})_4 = -\frac{\partial \mathcal{M}}{\partial \theta_1}\frac{\partial \mathcal{L}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{L}}{\partial \theta_1}\frac{\partial \mathcal{M}}{\partial t_{n\alpha}} - \frac{\partial \mathcal{V}}{\partial \theta_1}\frac{\partial \mathcal{U}}{\partial t_{n\alpha}} + \frac{\partial \mathcal{U}}{\partial \theta_1}\frac{\partial \mathcal{V}}{\partial t_{n\alpha}} = \frac{\partial \mathcal{B}_{n\alpha}}{\partial \theta_1},$$

which together with (4.4) yields

$$\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_{n\alpha}} \\ \frac{\partial \mathcal{M}}{\partial t_{n\alpha}} \\ \frac{\partial \mathcal{U}}{\partial t_{n\alpha}} \\ \frac{\partial \mathcal{V}}{\partial t_{n\alpha}} \end{pmatrix} = \mathbf{B} \begin{pmatrix} -\frac{\partial \mathcal{B}_{n\alpha}}{\partial x} \\ \frac{\partial \mathcal{B}_{n\alpha}}{\partial k} \\ -\frac{\partial \mathcal{B}_{n\alpha}}{\partial \theta_2} \\ \frac{\partial \mathcal{B}_{n\alpha}}{\partial \theta_1} \end{pmatrix} = \begin{pmatrix} \{\mathcal{B}_{n\alpha}, \mathcal{L}\} \\ \{\mathcal{B}_{n\alpha}, \mathcal{M}\} \\ \{\mathcal{B}_{n\alpha}, \mathcal{U}\} \\ \{\mathcal{B}_{n\alpha}, \mathcal{V}\} \end{pmatrix}.$$

□

Proposition 6. *Let $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ be a solution of the TdKP hierarchy. Then there exists a set of functions $(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})$ on the phase space such that equations (4.1) and (4.2) hold.*

Proof. Let $\varphi(t, x, \theta)$ be the dressing function for $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$. Set

$$f^{(1)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}(k), \qquad f^{(2)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}(x),$$
$$f^{(3)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}(\theta_1), \qquad f^{(4)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}(\theta_2)$$

then $f^{(i)}$ satisfy the Poisson relations (4.1). From the dressing form, we have

$$\mathcal{L}(t=0) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(k), \qquad \mathcal{M}(t=0) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(x),$$
$$\mathcal{U}(t=0) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(\theta_1), \qquad \mathcal{V}(t=0) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(\theta_2),$$

which implies

$$f^{(1)}(\mathcal{L}(t=0), \mathcal{M}(t=0), \mathcal{U}(t=0), \mathcal{V}(t=0)) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)} f^{(1)}(k, x, \theta) = k,$$
$$f^{(2)}(\mathcal{L}(t=0), \mathcal{M}(t=0), \mathcal{U}(t=0), \mathcal{V}(t=0)) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)} f^{(2)}(k, x, \theta) = x,$$
$$f^{(3)}(\mathcal{L}(t=0), \mathcal{M}(t=0), \mathcal{U}(t=0), \mathcal{V}(t=0)) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)} f^{(3)}(k, x, \theta) = \theta_1,$$
$$f^{(4)}(\mathcal{L}(t=0), \mathcal{M}(t=0), \mathcal{U}(t=0), \mathcal{V}(t=0)) = \mathrm{e}^{\mathrm{ad}\varphi(t=0)} f^{(4)}(k, x, \theta) = \theta_2.$$

Now one can verify the functional equations $f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = 0$ $(i = 1, 2, 3, 4.)$ from the initial value problem by using the Lax equations (2.2). For $f^{(1)}$ we have $\partial_{n\alpha} f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V}) = \{\mathcal{B}_{n\alpha}, f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})\}$. Hence

$$\partial_{n\alpha} f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})|_{t=0} = \{\mathcal{B}_{n\alpha}(t=0), k\},$$

which only contains non-negative powers of k. By induction, one can show that the coefficients of the Taylor expansion at $t = 0$, i.e. $\partial_{n_1\alpha_1} \cdots \partial_{n_j\alpha_j} f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})|_{t=0}$, will only contain non-negative powers of k for any multi-index $(n_1\alpha_1, \ldots, n_j\alpha_j)$. This completes the proof for the existence of $f^{(1)}$ for the constraint $f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = 0$. The other three constraints can be proved in a similar way. □

5. Additional symmetries

Having established the twistor construction for the solution space of the TdKP hierarchy one may investigate the symmetries of the solution space. Since these symmetries commute with hierarchy flows but do not commute between themselves, they were referred to additional symmetries [25]. In particular, we shall investigate how the dressing function φ changes under such symmetries.

Consider an infinitesimal transformation of the twistor data

$$f^{(i)}(k, x, \theta) \mapsto f_\epsilon^{(i)}(k, x, \theta) = \mathrm{e}^{-\epsilon \operatorname{ad} F} f^{(i)}(k, x, \theta), \qquad i = 1, 2, 3, 4 \tag{5.1}$$

where $F = F(k, x, \theta_1, \theta_2)$ and

$$\operatorname{ad} F = \frac{\partial F}{\partial k}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial k} + \frac{\partial F}{\partial \theta_1}\frac{\partial}{\partial \theta_2} - \frac{\partial F}{\partial \theta_2}\frac{\partial}{\partial \theta_1}.$$

Equation (5.1) can be regarded as a canonical transformation since it preserves the canonical relation (4.1) Denoting the corresponding transformation of $(\varphi, \mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ as

$$\begin{aligned} &\varphi \mapsto \varphi_\epsilon = \varphi + \epsilon \delta_F \varphi, \\ &\mathcal{L} \mapsto \mathcal{L}_\epsilon = \mathcal{L} + \epsilon \delta_F \mathcal{L}, \qquad \mathcal{M} \mapsto \mathcal{M}_\epsilon = \mathcal{M} + \epsilon \delta_F \mathcal{M}, \\ &\mathcal{U} \mapsto \mathcal{U}_\epsilon = \mathcal{U} + \epsilon \delta_F \mathcal{U}, \qquad \mathcal{V} \mapsto \mathcal{V}_\epsilon = \mathcal{V} + \epsilon \delta_F \mathcal{V}, \end{aligned}$$

where the infinitesimal transformation δ_F has no effect on the time variables, i.e. $\delta_F t_{n\alpha} = 0$.

Lemma 7. *Let $\mathcal{P}^{(i)}(k, x, t, \theta) \in \Lambda_{\geqslant 0}$ and satisfy the canonical Poisson relations*

$$\begin{aligned} &\{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\} = \{\mathcal{P}^{(3)}, \mathcal{P}^{(4)}\} = 1, \\ &\{\mathcal{P}^{(i)}, \mathcal{P}^{(j)}\} = 0, \qquad \text{otherwise.} \end{aligned} \tag{5.2}$$

If $\mathcal{A}(k, x, t, \theta)$ Poisson commute with $\mathcal{P}^{(i)}$ up to positive powers of k, namely,

$$\{\mathcal{A}, \mathcal{P}^{(i)}\}_{\leqslant -1} = 0, \qquad i = 1, 2, 3, 4$$

then $\mathcal{A}_{\leqslant -1} = 0$.

Proof. The proof is basically the same as that for dKP [34]. Let

$$\mathcal{Q}^{(i)} = \{\mathcal{A}, \mathcal{P}^{(i)}\} = \frac{\partial \mathcal{A}}{\partial k}\frac{\partial \mathcal{P}^{(i)}}{\partial x} - \frac{\partial \mathcal{A}}{\partial x}\frac{\partial \mathcal{P}^{(i)}}{\partial k} + \frac{\partial \mathcal{A}}{\partial \theta_1}\frac{\partial \mathcal{P}^{(i)}}{\partial \theta_2} - \frac{\partial \mathcal{A}}{\partial \theta_2}\frac{\partial \mathcal{P}^{(i)}}{\partial \theta_1}$$

or in matrix form

$$\begin{pmatrix} \mathcal{Q}^{(1)} \\ \mathcal{Q}^{(2)} \\ \mathcal{Q}^{(3)} \\ \mathcal{Q}^{(4)} \end{pmatrix} = \mathbf{D} \begin{pmatrix} -\frac{\partial \mathcal{A}}{\partial x} \\ \frac{\partial \mathcal{A}}{\partial k} \\ -\frac{\partial \mathcal{A}}{\partial \theta_2} \\ \frac{\partial \mathcal{A}}{\partial \theta_1} \end{pmatrix}, \tag{5.3}$$

where the matrix elements of $\mathbf{D}$ can be constructed from those of $\mathbf{B}$ by replacing $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ by $(\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \mathcal{P}^{(3)}, \mathcal{P}^{(4)})$. Due to the symplectic condition (5.2), $\mathbf{D}$ is a symplectic matrix as well and thus has an inverse $\mathbf{D}^{-1} = -\mathbf{J}\mathbf{D}^t\mathbf{J}$ which contains only non-negative powers of k. Hence (5.3) can be written as

$$\begin{pmatrix} -\frac{\partial\mathcal{A}}{\partial x} \\ \frac{\partial\mathcal{A}}{\partial k} \\ -\frac{\partial\mathcal{A}}{\partial\theta_2} \\ \frac{\partial\mathcal{A}}{\partial\theta_1} \end{pmatrix} = \mathbf{D}^{-1} \begin{pmatrix} \mathcal{Q}^{(1)} \\ \mathcal{Q}^{(2)} \\ \mathcal{Q}^{(3)} \\ \mathcal{Q}^{(4)} \end{pmatrix},$$

which implies that $\left(-\frac{\partial\mathcal{A}}{\partial x}, \frac{\partial\mathcal{A}}{\partial k}, -\frac{\partial\mathcal{A}}{\partial\theta_2}, \frac{\partial\mathcal{A}}{\partial\theta_1}\right)$ do not contain negative powers of k. This means that $\mathcal{A}_{\leqslant -1} = 0$. □

Proposition 8. *The changes of* $(\varphi, \mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ *under the infinitesimal symmetries defined by (5.1) are given by*

$$\nabla_{\delta_F\varphi}\varphi = F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}, \tag{5.4}$$

$$\delta_F\mathcal{L} = \{F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}, \mathcal{L}\}, \qquad \delta_F\mathcal{M} = \{F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}, \mathcal{M}\}, \tag{5.5}$$

$$\delta_F\mathcal{U} = \{F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}, \mathcal{U}\}, \qquad \delta_F\mathcal{V} = \{F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}, \mathcal{V}\}. \tag{5.6}$$

Proof. Here we only prove the case for the dressing function φ since equations (5.5) and (5.6), via dressing approach, are a direct consequence of (5.4). Under the transformation (5.1) the Poisson brackets (4.1) are still maintained and the constraints (4.2) become

$$\left[f_\epsilon^{(i)}(\mathcal{L}_\epsilon, \mathcal{M}_\epsilon, \mathcal{U}_\epsilon, \mathcal{V}_\epsilon)\right]_{\leqslant -1} = 0. \tag{5.7}$$

Since

$$\begin{aligned} f_\epsilon^{(i)}(\mathcal{L}_\epsilon, \mathcal{M}_\epsilon, \mathcal{U}_\epsilon, \mathcal{V}_\epsilon) &= \left(1 + \epsilon\,\mathrm{ad}\nabla_{\delta_F\varphi}\varphi\right) \mathrm{e}^{\mathrm{ad}\varphi}\, \mathrm{e}^{\mathrm{ad}t(k,\theta_1)}(1 - \epsilon\,\mathrm{ad}F) f^{(i)}(k, x, \theta), \\ &= f^{(i)} + \epsilon\left\{\nabla_{\delta_F\varphi}\varphi - F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V}), f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})\right\}. \end{aligned}$$

Hence (5.7) together with (4.2) implies

$$\left\{\nabla_{\delta_F\varphi}\varphi - F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V}), f^{(i)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})\right\}_{\leqslant -1} = 0$$

from which, by lemma 7, one gets

$$\nabla_{\delta_F\varphi}\varphi = F(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1}.$$

□

Proposition 9. *For any two generating functions* $F_1(k, x, \theta_1, \theta_2)$ *and* $F_2(k, x, \theta_1, \theta_2)$ *the corresponding infinitesimal symmetries* δ_{F_1} *and* δ_{F_2} *obey the commutation relations*

$$\left[\delta_{F_1}, \delta_{F_2}\right]\mathcal{K} = \delta_{\{F_1, F_2\}}\mathcal{K}, \tag{5.8}$$

where $\mathcal{K} = \varphi, \mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V}$.

Proof. First let us prove the case for $\mathcal{K} = \varphi$. Denoting $\delta_1 = \delta_{F_1}$ and $F_{1-} = (F_1)_{\leqslant -1}$ we get

$$\nabla_{\delta_1\varphi}\varphi = \frac{\mathrm{e}^{\mathrm{ad}\varphi} - 1}{\mathrm{ad}\varphi}\delta_1\varphi = F_{1-}$$

or

$$(\mathrm{e}^{\mathrm{ad}\varphi} - 1)\delta_1\varphi = \{\varphi, F_{1-}\}.$$

Taking the variation δ_2 we have

$$(\mathrm{e}^{\mathrm{ad}\varphi} - 1)(\delta_2\delta_1\varphi) = \{\varphi, (\delta_2 F_1)_-\} - \{F_{2-}, \{\varphi, F_{1-}\}\} - \{F_{2-}, \delta_1\varphi\} - \{F_{1-}, \delta_2\varphi\}.$$

Interchanging the indices $1 \leftrightarrow 2$ and subtracting the above equation one obtains

$$
\begin{aligned}
(\mathrm{e}^{\mathrm{ad}\varphi} - 1)[\delta_1, \delta_2]\varphi &= \{\varphi, (\delta_1 F_2)_-\} - \{\varphi, (\delta_2 F_1)_-\} + \{F_{2-}, \{\varphi, F_{1-}\}\} - \{F_{1-}, \{\varphi, F_{2-}\}\},\\
&= \{\varphi, \{F_1, F_2\}_-\},\\
&= \mathrm{ad}\varphi\{F_1, F_2\}_-,
\end{aligned}
$$

which implies

$$
\left[\delta_{F_1}, \delta_{F_2}\right]\varphi = \delta_{\{F_1, F_2\}}\varphi.
$$

For $\mathcal{K} = \mathcal{L}$

$$
\begin{aligned}
[\delta_1, \delta_2]\mathcal{L} &= \{\{F_{1-}, F_2\}_-, \mathcal{L}\} + \{F_{2-}, \{F_{1-}, \mathcal{L}\}\} - (1 \leftrightarrow 2),\\
&= \{\{F_{1-}, F_2\}_-, \mathcal{L}\} - \{\{F_{2-}, F_1\}_-, \mathcal{L}\} - \{\{F_{1-}, F_{2-}\}, \mathcal{L}\},\\
&= \{\{F_1, F_2\}_-, \mathcal{L}\} = \delta_{\{F_1, F_2\}}\mathcal{L}.
\end{aligned}
$$

For $\mathcal{K} = \mathcal{M}, \mathcal{U}$ and $\mathcal{V}$ one can prove them in a similar way. □

We remark that $F(k, x, \theta)$, the generating function of additional symmetries, can be expressed in Laurent–Fourier series of the form

$$
F(k, x, \theta) = \sum_{m,n,\alpha,\beta} c_{mn}^{\alpha\beta} k^m x^n \,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\,\mathrm{e}^{\mathrm{i}\beta\theta_2},
$$

where $c_{mn}^{\alpha\beta}$ are constants. For the generator $k^m x^n \,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\,\mathrm{e}^{\mathrm{i}\beta\theta_2}$ we identify the associated infinitesimal derivative $\delta_{k^m x^n \mathrm{e}^{\mathrm{i}\alpha\theta_1}\mathrm{e}^{\mathrm{i}\beta\theta_2}} := Q_{mn}^{\alpha\beta}$ and then the commutation relation (5.8) can be realized as a Lie algebra of Poisson brackets [2]

$$
\left[Q_{mn}^{\alpha\beta}, Q_{m'n'}^{\alpha'\beta'}\right] = (mn' - nm')Q_{m+m'-1,n+n'-1}^{\alpha+\alpha',\beta+\beta'} - (\alpha\beta' - \beta\alpha')Q_{m+m',n+n'}^{\alpha+\alpha',\beta+\beta'}.
$$

6. Gelfand–Dickey reductions

Let us consider the following twistor data $f^{(i)}$ $(i = 1, 2, 3, 4)$ for the TdKP hierarchy

$$
f^{(1)} = k^N, \qquad f^{(2)} = \frac{xk^{1-N}}{N}, \qquad f^{(3)} = \theta_1, \qquad f^{(4)} = \theta_2, \qquad (N \geqslant 2) \tag{6.1}
$$

which satisfy the Poisson relations (4.1). Then the constraints (4.2) imply that

$$
f^{(1)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = \mathcal{L}^N_{\leqslant -1} = 0, \tag{6.2}
$$

$$
f^{(2)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = (\mathcal{M}\mathcal{L}^{1-N})_{\leqslant -1} = 0, \tag{6.3}
$$

$$
f^{(3)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = \mathcal{U}_{\leqslant -1} = 0, \tag{6.4}
$$

$$
f^{(4)}(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})_{\leqslant -1} = \mathcal{V}_{\leqslant -1} = 0. \tag{6.5}
$$

First, from (6.4), we observe that $\mathcal{U} = \theta_1$, i.e. $u_n = 0$, $\forall n$. Therefore the commutation relations $\{\mathcal{L}, \mathcal{U}\} = \{\mathcal{M}, \mathcal{U}\} = 0$ and $\{\mathcal{U}, \mathcal{V}\} = 1$ imply that $\partial g_n/\partial\theta_2 = \partial h_n/\partial\theta_2 = \partial v_n/\partial\theta_2 = 0$. Next from equation (6.2) one can define an Nth-order Lax operator of the form

$$
L = (\mathcal{L}^N)_{\geqslant 0} = k^N + a_2 k^{N-2} + \cdots + a_N, \tag{6.6}
$$

which satisfies the Lax equation

$$
\frac{\partial L}{\partial t_{n\alpha}} = \{(L^{n/N}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1})_{\geqslant 0}, L\}.
$$

Due to the fact that $\partial L/\partial\theta_2 = 0$ we have $\partial L/\partial t_{jN,\alpha} = 0$, $j = 0, 1, 2, \ldots$. Hence the variables a_i do not depend on $t_{0,\alpha}, t_{N,\alpha}, t_{2N,\alpha}, \ldots$. From (6.3) we have

$$0 = \sum_{n=1}^{N-1}\sum_{\alpha} nt_{n\alpha}\mathcal{L}^{n-N}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=N+1}^{\infty}\sum_{\alpha} nt_{n\alpha}(\mathcal{L}^{n-N})_{\leqslant -1}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + x\mathcal{L}^{1-N} + \sum_{n=1} h_n\mathcal{L}^{-n-N}.$$

Now multiplying $\mathcal{L}^{N-j-1}$ and using the formula $\mathrm{res}(\mathcal{L}^n d_k\mathcal{L}) = \delta_{n,-1}$ we get the hodograph equations:

$$0 = \sum_{\alpha} t_{j\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=1}\sum_{\alpha} t_{N+n,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\mu_n^j(a), \qquad 1 \leqslant j \leqslant N-1, \tag{6.7}$$

where the hodograph coefficients $\mu_n^j(a)$ are defined by

$$\mu_n^j(a) = \frac{N+n}{j}\,\mathrm{res}[\mathcal{L}^{N-j-1}(\mathcal{L}^n)_{\leqslant -1}d_k\mathcal{L}]. \tag{6.8}$$

Similarly, multiplying (6.3) and (6.5) by $\mathcal{L}^{N+j-1}$ and $\mathcal{L}^{j-1}$ respectively, we have

$$0 = h_j + \sum_{n=N+1}^{\infty}\sum_{\alpha}\mathrm{e}^{\mathrm{i}\alpha\theta_1} nt_{n\alpha}\,\mathrm{res}[\mathcal{L}^{N+j-1}(\mathcal{L}^{n-N})_{\leqslant -1}d_k\mathcal{L}], \qquad j = 1, 2, \ldots,$$

$$0 = v_j + \sum_{n=1}^{\infty}{\sum_{\alpha}}' \mathrm{i}\alpha t_{n\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\,\mathrm{res}[\mathcal{L}^{j-1}(\mathcal{L}^n)_{\leqslant -1}d_k\mathcal{L}], \qquad j = 1, 2, \ldots,$$

where $\sum_\alpha'$ denotes that the term $\alpha = 0$ has been omitted. The above equations show that h_j and v_j can be expressed in terms of a_i.

Remark 2. Note that the twistor data (6.1) can be deformed by adding $C(k)$, an arbitrary function of k, to $f^{(2)}$ [30, 34]. As a consequence, the hodograph equation now becomes

$$0 = \sum_{\alpha} t_{j\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=1}\sum_{\alpha} t_{N+n,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\mu_n^j(a) + N\,\mathrm{res}(\mathcal{L}^{N-j-1}C(\mathcal{L})d_k\mathcal{L}), \tag{6.9}$$

where $1 \leqslant j \leqslant N-1$. If $C(k)$ has a Laurent expansion $C(k) = \sum_n C_n k^n$ where C_n are constants. Then the last term in (6.9) has the form $\sum_{n=1}\frac{Nj}{N+n}C_n\mu_n^j$ which readily implies that the coefficients of the deformation $C(k)$ can be absorbed into time variables as a shift $t_{N+n,0} \mapsto t_{N+n,0} + \frac{Nj}{N+n}C_n$.

Let us illustrate hodograph solutions for two simple examples.

Example 1. $N = 2$. In this case

$$L = (\mathcal{L}^2)_{\geqslant 0} = \mathcal{L}^2 = k^2 + a_2,$$

which gives a one-variable reduction of the TdKP hierarchy ($a_2 = 2g_2$) defined by the Lax equation

$$\frac{\partial L}{\partial t_{2j+1,\alpha}} = \{(L^{2j+1/2}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1})_{\geqslant 0}, L\}, \qquad j = 0, 1, 2, \ldots,$$

or

$$\frac{\partial a_2}{\partial t_{2j+1,\alpha}} = \mathrm{e}^{\mathrm{i}\alpha\theta_1}\frac{(2j+1)!!}{2^j j!}a_2^j\frac{\partial a_2}{\partial x}. \tag{6.10}$$

These equations are (1+1)-dimensional hydrodynamic-type equations and their solutions can be expressed as implicit functions through the hodograph equation (6.7):

$$0 = \sum_{\alpha} t_{1\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=1}\sum_{\alpha} t_{2n+1,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\mu_n^1(a), \tag{6.11}$$

where

$$\mu_n^1(a) = (n+2)\,\mathrm{res}[(\mathcal{L}^{2n-1})_{\leqslant -1} d_k \mathcal{L}].$$

If we impose the condition $t_{n,\alpha} = 0, n > 3$ for the time variables, then the hodograph equation (6.11) yields

$$a_2(x,t,\theta) = -\frac{2}{3}\left(\frac{x + \sum'_\alpha t_{1\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}{t_{30} + \sum'_\alpha t_{3\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}\right)$$

which solves (6.10) for $j = 0, 1$.

Example 2. $N = 3$. In this case

$$L = (\mathcal{L}^3)_{\geqslant 0} = \mathcal{L}^3 = k^3 + a_2 k + a_3,$$

which obeys the Lax equation

$$\frac{\partial L}{\partial t_{k\alpha}} = \{(L^{k/3}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1})_{\geqslant 0}, L\},$$

or

$$\begin{aligned}\frac{\partial a_2}{\partial t_{n\alpha}} &= 3\sum_{j=1}^{[\frac{n+1}{2}]}\binom{\frac{n}{3}}{j}\binom{j}{n-2j+1}\mathrm{e}^{\mathrm{i}\alpha\theta_1}\left(a_2^{3j-n-1}a_3^{n-2j+1}\right)_x,\\ \frac{\partial a_3}{\partial t_{n\alpha}} &= 3\sum_{j=1}^{[\frac{n+2}{2}]}\binom{\frac{n}{3}}{j}\binom{j}{n-2j+2}\mathrm{e}^{\mathrm{i}\alpha\theta_1}\left(a_2^{3j-n-2}a_3^{n-2j+2}\right)_x.\end{aligned} \tag{6.12}$$

This is a two-variable reduction of the TdKP hierarchy ($a_2 = 3g_2$ and $a_3 = 3g_3$). The solutions of these (1+1) hydrodynamic-type equations can be expressed as implicit functions through the hodograph equation (6.7):

$$\begin{aligned}0 &= \sum_\alpha t_{1\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=1}\sum_\alpha t_{3+n,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\mu_n^1(a),\\ 0 &= \sum_\alpha t_{2\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} + \sum_{n=1}\sum_\alpha t_{3+n,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}\mu_n^2(a),\end{aligned} \tag{6.13}$$

where $\mu_n^j(a)$ are defined by (6.8). If we impose the condition $t_{n,\alpha} = 0, n > 4$ for the time variables, then the hodograph equation (6.13) yields

$$\begin{aligned}a_2(x,t,\theta) &= -\frac{3}{2}\left(\frac{t_{20} + \sum'_\alpha t_{2\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}{t_{40} + \sum'_\alpha t_{4\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}\right),\\ a_3(x,t,\theta) &= -\frac{3}{4}\left(\frac{x + \sum'_\alpha t_{1\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}{t_{40} + \sum'_\alpha t_{4\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}}\right),\end{aligned}$$

which solves (6.12) for $n = 1, 2, 4$.

Since the above examples do not depend on θ_2 and thus θ_1 enters these solutions as a free parameter. To take θ_2 into account we shall properly choose $f^{(3)}$ (and hence $f^{(4)}$) instead of that provided in (6.1).

Let us consider another set of twistor data as follows:

$$\begin{aligned}&f^{(1)} = k^N, \qquad f^{(2)} = \frac{xk^{1-N}}{N} - \frac{\theta_1\theta_2 k^{-N}}{N},\\ &f^{(3)} = \theta_1 k, \qquad f^{(4)} = \theta_2 k^{-1}, \qquad (N \geqslant 2),\end{aligned}$$

which satisfy the Poisson relations (4.1) as well. Then the constraints (4.2) imply that

$$f^{(1)}(\mathcal{L},\mathcal{M},\mathcal{U},\mathcal{V})_{\leqslant -1} = \mathcal{L}^N_{\leqslant -1} = 0, \tag{6.14}$$

$$f^{(2)}(\mathcal{L},\mathcal{M},\mathcal{U},\mathcal{V})_{\leqslant -1} = (\mathcal{M}\mathcal{L}^{1-N} - \mathcal{U}\mathcal{V}\mathcal{L}^{-N})_{\leqslant -1} = 0, \tag{6.15}$$

$$f^{(3)}(\mathcal{L},\mathcal{M},\mathcal{U},\mathcal{V})_{\leqslant -1} = (\mathcal{U}\mathcal{L})_{\leqslant -1} = 0, \tag{6.16}$$

$$f^{(4)}(\mathcal{L},\mathcal{M},\mathcal{U},\mathcal{V})_{\leqslant -1} = (\mathcal{V}\mathcal{L}^{-1})_{\leqslant -1} = 0. \tag{6.17}$$

From the first constraint (6.14) we still have an Nth-order Lax operator of the form (6.6). From the third constraint (6.16), we have

$$\mathcal{U}\mathcal{L} = \theta_1 k + u_1, \tag{6.18}$$

which implies that u_n can be expressed in terms of g_j as

$$u_n = -\frac{\theta_1}{n-1}\,\mathrm{res}(\mathcal{L}^{n-1}\,\mathrm{d}k), \qquad n \geqslant 2.$$

where the residue formula was used. For instance,

$$u_2 = -\theta_1 g_2, \qquad u_3 = -\theta_1 g_3, \qquad u_4 = -\theta_1\big(g_4 + g_2^2\big), \quad \text{etc.}$$

For the coefficient u_1, using (6.18) and the canonical relation $\{\mathcal{L},\mathcal{U}\} = 0$, we have $0 = \{\mathcal{L}, \theta_1 k + u_1\}_{[0]}$ which results

$$u_1 = \partial_x^{-1}\partial_{\theta_2} g_2. \tag{6.19}$$

Next, from the fourth constraint (6.17), we have

$$0 = \sum_{n,\alpha} \mathrm{i}\alpha t_{n\alpha}(\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1} + \theta_2\mathcal{L}^{-1} + \sum_{n=1}^{\infty} v_n(t,x,\theta)\mathcal{L}^{-n-1},$$

which yields

$$\begin{aligned} v_j &= -\sum_{n,\alpha} \mathrm{i}\alpha t_{n\alpha}\,\mathrm{res}((\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}\mathcal{L}^j d_k\mathcal{L}), \qquad j \geqslant 1, \\ \theta_2 &= -\sum_{n,\alpha} \mathrm{i}\alpha t_{n\alpha}\,\mathrm{res}(\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}\,\mathrm{d}k). \end{aligned} \tag{6.20}$$

Finally, from the second constraint (6.15), we have

$$0 = \sum_{n,\alpha} n t_{n,\alpha}(\mathcal{L}^{n-N}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1} + x\mathcal{L}^{1-N} + \sum_{n=1}^{\infty} h_n(t,x,\theta)\mathcal{L}^{-n-N} - (\mathcal{U}\mathcal{V}\mathcal{L}^{1-N})_{\leqslant -1}. \tag{6.21}$$

In the following, we shall show that we can extract hodograph equations from (6.20) and (6.21) by setting $\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}} = \mathrm{e}^{\mathrm{i}\alpha\theta_1}\sum_{l=0}^{\infty} P_l(\mathrm{i}\alpha\mathbf{u})\mathcal{L}^{-l}$, where $\mathbf{u} = (u_1, u_2, \ldots)$ and $P_l(\mathbf{x})$ are the Schur polynomials defined by

$$P_0(\mathbf{x}) = 1, \qquad P_1(\mathbf{x}) = x_1, \qquad P_2(\mathbf{x}) = x_2 + \frac{x_1^2}{2!}, \qquad P_3(\mathbf{x}) = x_3 + x_1x_2 + \frac{x_1^3}{3!},$$

etc. with $\mathbf{x} = (x_1, x_2, \ldots)$. Multiplying $\mathcal{L}^{j-1}$ with $j = -(n-N-l)$ on (6.21) and using $\mathrm{res}(\mathcal{L}^n d_k\mathcal{L}) = \delta_{n,-1}$ again, we obtain

$$\begin{aligned} 0 = {}& \sum_{\alpha}\sum_{n=0}^{N-1}(n - \mathrm{i}\alpha\theta_1)\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} P_{j+n-N} t_{n\alpha} \\ &+ x\delta_{j,N-1} - \theta_1\theta_2\delta_{j,N} - \sum_{\alpha}\sum_{n=0}^{N-1}\sum_{m=1}^{j+n-N} \mathrm{i}\alpha\,\mathrm{e}^{\mathrm{i}\alpha\theta_1} u_m P_{j+n-m-N} t_{n\alpha} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}(h_n-\theta_1 v_n-\theta_2 u_n)\delta_{j,n+N}-\sum_{n,m=1}^{\infty} v_n u_m \delta_{j,n+m+N}\\
&+\sum_{\alpha}\sum_{n=0}^{\infty}(n+N-\mathrm{i}\alpha\theta_1)t_{n+N,\alpha}\,\mathrm{res}(\mathcal{L}^{j-1}(\mathcal{L}^n\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}d_k\mathcal{L})\\
&-\sum_{\alpha}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\mathrm{i}\alpha u_m t_{n+N,\alpha}\,\mathrm{res}(\mathcal{L}^{j-1}(\mathcal{L}^{n-m}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}d_k\mathcal{L}), \qquad j\geqslant 1.
\end{aligned}
\tag{6.22}
$$

Similarly, from (6.20), we have

$$
\theta_2+\sum_{\alpha}\mathrm{i}\alpha t_{0\alpha}\,\mathrm{e}^{\mathrm{i}\alpha\theta_1}=-\sum_{\alpha}\sum_{n=1}^{\infty}\mathrm{i}\alpha t_{n\alpha}\,\mathrm{res}((\mathcal{L}^{n-1}\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}d_k\mathcal{L}). \tag{6.23}
$$

Equation (6.22) together with (6.23) gives us hodograph solutions involving variables t, x, θ implicitly. Since the above hodograph equations are more complicated than those of the previous examples, we decide to solve (6.22) and (6.23) for the first few hierarchy flows by restricting $n \leqslant 3$ and $\alpha = 0, 1$. Let us illustrate the simplest case.

Example 3. $N = 2$. In this case

$$L=\mathcal{L}^2_{\geqslant 0}=\mathcal{L}^2=k^2+2g_2=k^2+a_2.$$

Also, from the formula $u_{n+1}=-\frac{\theta_1}{n}\,\mathrm{res}(\mathcal{L}^n\,\mathrm{d}k)$ for $n\geqslant 1$, we have

$$u_{2k}=-\frac{\theta_1}{2k-1}\,\mathrm{res}\big(L^{\frac{2k-1}{2}}\big)=-\frac{\theta_1}{2k-1}\binom{k-1/2}{k}a_2^k, \qquad k\geqslant 1$$

and $u_{2k+1}=0$ except $u_1=\partial_x^{-1}\partial_{\theta_2}g_2$. The Lax flow is given by

$$\partial_{n\alpha}L=\{(\mathcal{L}^n\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\geqslant 0}, L\}.$$

Extracting the zeroth-order term, one gets

$$
\begin{aligned}
\partial_{n\alpha}a_2&=\{(\mathcal{L}^n\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\geqslant 0}, k^2+a_2\}_{[0]},\\
&=\{k^2+a_2, (\mathcal{L}^n\,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\leqslant -1}\}_{[0]},\\
&=2\mathrm{e}^{\mathrm{i}\alpha\theta_1}\left(\sum_{j=0}^{[\frac{n+1}{2}]}\binom{j-1/2}{j}P_{n-2j+1}(\mathrm{i}\alpha\mathbf{u})a_2^j\right)_x .
\end{aligned}
\tag{6.24}
$$

Keeping the first few time variables $t_{01}, t_{10}, t_{11}, t_{21}$ and t_{30} only, then the hodograph equations (6.22) for $j = 1$ and (6.23) imply

$$
\begin{aligned}
x+t_{10}&=-3g_2t_{30}+\mathrm{i}(\mathrm{i}+\theta_1)\,\mathrm{e}^{\mathrm{i}\theta_1}t_{11}-(\mathrm{i}+\theta_1)\,\mathrm{e}^{\mathrm{i}\theta_1}u_1t_{21},\\
\theta_2+\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_1}t_{01}&=\mathrm{e}^{\mathrm{i}\theta_1}\big(u_1t_{11}+\mathrm{i}u_1^2t_{21}/2-(\mathrm{i}+\theta_1)g_2t_{21}\big),
\end{aligned}
$$

which, after some algebras, gives

$$
\begin{aligned}
u_1(x,t,\theta)&=\mathrm{i}\frac{t_{11}}{t_{21}}+\mathrm{i}(\mathrm{i}+\theta_1)^2\,\mathrm{e}^{\mathrm{i}\theta_1}\frac{t_{21}}{3t_{30}}\\
&\quad\pm\frac{\mathrm{i}}{3t_{21}t_{30}}\sqrt{(\mathrm{i}+\theta_1)^4\,\mathrm{e}^{2\mathrm{i}\theta_1}t_{21}^4+f+18\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\theta_1}(\theta_2+\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_1}t_{01})t_{21}t_{30}^2},\\
g_2(x,t,\theta)&=-\frac{x+t_{10}}{3t_{30}}-\mathrm{i}(\mathrm{i}+\theta_1)^3\,\mathrm{e}^{2\mathrm{i}\theta_1}\frac{t_{21}^2}{9t_{30}^2}\\
&\quad\mp\frac{\mathrm{i}(\mathrm{i}+\theta_1)}{9t_{30}^2}\sqrt{(\mathrm{i}+\theta_1)^4\,\mathrm{e}^{4\mathrm{i}\theta_1}t_{21}^4+\mathrm{e}^{2\mathrm{i}\theta_1}f+18\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_1}(\theta_2+\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_1}t_{01})t_{21}t_{30}^2},
\end{aligned}
$$

where

$$f = 9t_{11}^2 t_{30}^2 + 6(1 - i\theta_1)(x + t_{10})t_{21}^2 t_{30}.$$

One can verify that $a_2 = 2g_2$ satisfies the hierarchy flow (6.24) as well as the constraint (6.19).

Example 4. $N = 3$. In this case

$$L = \mathcal{L}^3_{\geqslant 0} = \mathcal{L}^3 = k^3 + 3g_2 k + 3g_3 = k^3 + a_2 k + a_3$$

and

$$u_{n+1} = -\frac{\theta_1}{n} \sum_{j=0}^{[\frac{n+1}{2}]} \binom{n/3}{j} \binom{j}{n+1-2j} a_2^{3j-n-1} a_3^{n+1-2j}, \qquad n \geqslant 1.$$

The hierarchy flow of a_2 and a_3 are given by

$$\partial_{n\alpha} a_2 = 3e^{i\alpha\theta_1} \left(\sum_{l=0}^{n+1} P_l(i\alpha\mathbf{u}) \sum_{j=0}^{[\frac{n-l+1}{2}]} \binom{\frac{n-l}{3}}{j} \binom{j}{n-l+1-2j} a_2^{3j-n+l-1} a_3^{n-l+1-2j} \right)_x,$$

$$\partial_{n\alpha} a_3 = 3e^{i\alpha\theta_1} \left(\sum_{l=0}^{n+2} P_l(i\alpha\mathbf{u}) \sum_{j=0}^{[\frac{n-l+2}{2}]} \binom{\frac{n-l}{3}}{j} \binom{j}{n-l+2-2j} a_2^{3j-n+l-2} a_3^{n-l+2-2j} \right)_x$$
$$+ \left\{ a_2, e^{i\alpha\theta_1} \sum_{l=0}^{n+1} P_l(i\alpha\mathbf{u}) \sum_{j=0}^{[\frac{n-l+1}{2}]} \binom{\frac{n-l}{3}}{j} \binom{j}{n-l+1-2j} a_2^{3j-n+l-1} a_3^{n-l+1-2j} \right\}_\theta.$$

Since the computation of hodograph solutions, which can be obtained from (6.22) and (6.23), is quite involved, we leave the details to those interested readers.

We would like to stress that the hodograph solutions of these finite-dimensional reductions are not double periodic. Perhaps, more clever twistor functions have to be chosen.

7. Miura transformation

In this section, we like to consider another higher-dimensional system defined by the Lie algebraic decomposition for $l = 1$ with respect to the Poisson bracket (2.1). A convenient way for introducing this system is to construct a Miura transformation to the TdKP.

Proposition 10. *Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$ be related to $(\mathcal{L}, \mathcal{M}, \mathcal{U}, \mathcal{V})$ as*

$$\tilde{\mathcal{L}} = e^{\mathrm{ad}\xi}(\mathcal{L}), \qquad \tilde{\mathcal{M}} = e^{\mathrm{ad}\xi}(\mathcal{M}), \qquad \tilde{\mathcal{U}} = e^{\mathrm{ad}\xi}(\mathcal{U}), \qquad \tilde{\mathcal{V}} = e^{\mathrm{ad}\xi}(\mathcal{V}) \tag{7.1}$$

where the dressing function $\xi = \xi(x, t, \theta)$ is defined by

$$\nabla_{t_{n\alpha},\xi}\xi = -(e^{\mathrm{ad}\xi}(\mathcal{L}^n e^{i\alpha\mathcal{U}}))_{[0]}.$$

Then the Laurent series $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$ satisfy the Lax equations

$$\frac{\partial \tilde{\mathcal{L}}}{\partial t_{n\alpha}} = \{\tilde{\mathcal{B}}_{n\alpha}, \tilde{\mathcal{L}}\}, \qquad \frac{\partial \tilde{\mathcal{M}}}{\partial t_{n\alpha}} = \{\tilde{\mathcal{B}}_{n\alpha}, \tilde{\mathcal{M}}\},$$
$$\frac{\partial \tilde{\mathcal{U}}}{\partial t_{n\alpha}} = \{\tilde{\mathcal{B}}_{n\alpha}, \tilde{\mathcal{U}}\}, \qquad \frac{\partial \tilde{\mathcal{V}}}{\partial t_{n\alpha}} = \{\tilde{\mathcal{B}}_{n\alpha}, \tilde{\mathcal{V}}\} \tag{7.2}$$

and the canonical relations

$$\{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\} = \{\tilde{\mathcal{U}}, \tilde{\mathcal{V}}\} = 1, \qquad \{\tilde{\mathcal{L}}, \tilde{\mathcal{U}}\} = \{\tilde{\mathcal{L}}, \tilde{\mathcal{V}}\} = \{\tilde{\mathcal{M}}, \tilde{\mathcal{U}}\} = \{\tilde{\mathcal{M}}, \tilde{\mathcal{V}}\} = 0$$

where $\tilde{\mathcal{B}}_{n\alpha} = (\tilde{\mathcal{L}}^n \,\mathrm{e}^{\mathrm{i}\alpha\tilde{\mathcal{U}}})_{\geqslant 1}$.

Proof. Using lemma 2(a) we have

$$\begin{aligned}\frac{\partial \tilde{\mathcal{L}}}{\partial t_{n\alpha}} &- \{(\tilde{\mathcal{L}}^n \,\mathrm{e}^{\mathrm{i}\alpha\tilde{\mathcal{U}}})_{\geqslant 1}, \tilde{\mathcal{L}}\} \\ &= \mathrm{e}^{\mathrm{ad}\xi}\left(\frac{\partial \mathcal{L}}{\partial t_{n\alpha}}\right) + \left\{\nabla_{t_{n\alpha},\xi}\xi, \tilde{\mathcal{L}}\right\} - \{\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\geqslant 0} - (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{L}}\}, \\ &= \mathrm{e}^{\mathrm{ad}\xi}\left(\frac{\partial \mathcal{L}}{\partial t_{n\alpha}} - \{(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})_{\geqslant 0}, \mathcal{L}\}\right) + \left\{\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{L}}\right\},\end{aligned}$$

which yields the first equation of (7.2) if

$$\left\{\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{L}}\right\} = 0. \tag{7.3}$$

Similarly, the other equations of (7.2) hold if

$$\left\{\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{M}}\right\} = 0, \tag{7.4}$$

$$\left\{\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{U}}\right\} = 0, \tag{7.5}$$

$$\left\{\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}, \tilde{\mathcal{V}}\right\} = 0. \tag{7.6}$$

Let $\mathcal{A} = \mathrm{e}^{-\mathrm{ad}\xi}\left(\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]}\right)$, then (7.3)–(7.6) become

$$\{\mathcal{A}, \mathcal{L}\} = 0, \qquad \{\mathcal{A}, \mathcal{M}\} = 0, \qquad \{\mathcal{A}, \mathcal{U}\} = 0, \qquad \{\mathcal{A}, \mathcal{V}\} = 0.$$

which, in matrix form, can be expressed as

$$\mathbf{B}\begin{pmatrix} -\frac{\partial \mathcal{A}}{\partial x} \\ \frac{\partial \mathcal{A}}{\partial k} \\ -\frac{\partial \mathcal{A}}{\partial \theta_2} \\ \frac{\partial \mathcal{A}}{\partial \theta_1}\end{pmatrix} = 0.$$

Since $\mathbf{B}$ is a symplectic matrix, $\mathbf{B}^{-1}$ exists. This implies that $\partial\mathcal{A}/\partial x = \partial\mathcal{A}/\partial k = \partial\mathcal{A}/\partial\theta_1 = \partial\mathcal{A}/\partial\theta_2 = 0$. That means

$$\nabla_{t_{n\alpha},\xi}\xi + (\mathrm{e}^{\mathrm{ad}\xi}(\mathcal{L}^n \,\mathrm{e}^{\mathrm{i}\alpha\mathcal{U}}))_{[0]} = 0.$$

□

Equation (7.1) can be regarded as a higher-dimensional generalization of the Miura transformation between the dKP and dmKP hierarchies [5]. Hence we refer the set of equations (7.2) as TdmKP hierarchy. From the dressing operators (7.1), $\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}$, and $\tilde{\mathcal{V}}$ are the Laurent series of the form

$$\begin{aligned}\tilde{\mathcal{L}} &= k + \sum_{n=0}^{\infty} \tilde{g}_{n+1}(t, x, \theta) k^{-n}, \\ \tilde{\mathcal{M}} &= \sum_{n,\alpha} n t_{n\alpha} \tilde{\mathcal{L}}^{n-1} \,\mathrm{e}^{\mathrm{i}\alpha\tilde{\mathcal{U}}} + x + \sum_{n=1}^{\infty} \tilde{h}_n(t, x, \theta) \tilde{\mathcal{L}}^{-n-1}, \\ \tilde{\mathcal{U}} &= \theta_1 + \sum_{n=0}^{\infty} \tilde{u}_n(t, x, \theta) \tilde{\mathcal{L}}^{-n}, \\ \tilde{\mathcal{V}} &= \sum_{n,\alpha} \mathrm{i}\alpha t_{n\alpha} \tilde{\mathcal{L}}^{n} \,\mathrm{e}^{\mathrm{i}\alpha\tilde{\mathcal{U}}} + \theta_2 + \sum_{n=0}^{\infty} \tilde{v}_n(t, x, \theta) \mathcal{L}^{-n}.\end{aligned} \tag{7.7}$$

Proposition 11. *Given a set of functions* $(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)})$ *on the phase space* $(k, x, \theta_1, \theta_2)$ *which satisfy the Poisson relations*

$$\begin{aligned} &\{\tilde{f}^{(1)}, \tilde{f}^{(2)}\} = \{\tilde{f}^{(3)}, \tilde{f}^{(4)}\} = 1, \\ &\{\tilde{f}^{(i)}, \tilde{f}^{(j)}\} = 0, \qquad \text{otherwise}. \end{aligned} \tag{7.8}$$

If $\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}}$ *are Laurent series of the form (7.7) then the following conditions*

$$\begin{aligned} &\tilde{f}^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})_{\leqslant 0} = 0, \\ &\tilde{f}^{(i)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})_{\leqslant -1} = 0, \qquad i = 2, 3, 4. \end{aligned} \tag{7.9}$$

give a solution $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$ *of the TdmKP hierarchy. We call* $(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)})$ *the twistor data of the corresponding solution. Conversely, let* $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$ *be a solution of the TdmKP. Then there exists a set of functions* $(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)})$ *on the phase space such that equations (7.8) and (7.9) hold.*

Proof. The proof for the first part of the proposition is the same as the TdKP. Here we give a proof for the second part using the Miura transformation (7.1). Let us choose the twistor data for the TdmKP system as

$$\begin{aligned} &\tilde{f}^{(1)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}\, \mathrm{e}^{-\mathrm{ad}\xi(t=0)}(k), \qquad \tilde{f}^{(2)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}\, \mathrm{e}^{-\mathrm{ad}\xi(t=0)}(x), \\ &\tilde{f}^{(3)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}\, \mathrm{e}^{-\mathrm{ad}\xi(t=0)}(\theta_1), \qquad \tilde{f}^{(4)}(k, x, \theta) = \mathrm{e}^{-\mathrm{ad}\varphi(t=0)}\, \mathrm{e}^{-\mathrm{ad}\xi(t=0)}(\theta_2); \end{aligned}$$

then $\tilde{f}^{(i)}$ satisfy the Poisson relations (7.8). From the dressing form (3.2), (3.4), and (7.1), we have

$$\begin{aligned} &\tilde{\mathcal{L}}(t=0) = \mathrm{e}^{\mathrm{ad}\xi(t=0)}\, \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(k), \qquad \tilde{\mathcal{M}}(t=0) = \mathrm{e}^{\mathrm{ad}\xi(t=0)}\, \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(x), \\ &\tilde{\mathcal{U}}(t=0) = \mathrm{e}^{\mathrm{ad}\xi(t=0)}\, \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(\theta_1), \qquad \tilde{\mathcal{V}}(t=0) = \mathrm{e}^{\mathrm{ad}\xi(t=0)}\, \mathrm{e}^{\mathrm{ad}\varphi(t=0)}(\theta_2), \end{aligned}$$

which implies

$$\begin{aligned} &\tilde{f}^{(1)}(\tilde{\mathcal{L}}(t=0), \tilde{\mathcal{M}}(t=0), \tilde{\mathcal{U}}(t=0), \tilde{\mathcal{V}}(t=0)) = k, \\ &\tilde{f}^{(2)}(\tilde{\mathcal{L}}(t=0), \tilde{\mathcal{M}}(t=0), \tilde{\mathcal{U}}(t=0), \tilde{\mathcal{V}}(t=0)) = x, \\ &\tilde{f}^{(3)}(\tilde{\mathcal{L}}(t=0), \tilde{\mathcal{M}}(t=0), \tilde{\mathcal{U}}(t=0), \tilde{\mathcal{V}}(t=0)) = \theta_1, \\ &\tilde{f}^{(4)}(\tilde{\mathcal{L}}(t=0), \tilde{\mathcal{M}}(t=0), \tilde{\mathcal{U}}(t=0), \tilde{\mathcal{V}}(t=0)) = \theta_2. \end{aligned}$$

Now one can verify the functional equations (7.9) from the initial value problem by using the Lax equations (7.2). For $\tilde{f}^{(1)}$ we have $\partial_{n\alpha}\tilde{f}^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}}) = \{\tilde{\mathcal{B}}_{n\alpha}, \tilde{f}^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})\}$. Hence

$$\partial_{n\alpha}\tilde{f}^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})|_{t=0} = \{\tilde{\mathcal{B}}_{n\alpha}(t=0), k\},$$

which only contains power $\geqslant 1$ of k. By induction, one can show that the coefficients of Taylor expansion at $t = 0$, i.e., $\partial_{n_1\alpha_1}\cdots\partial_{n_j\alpha_j}\tilde{f}^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})|_{t=0}$, only contain power $\geqslant 1$ of k for any multi-index $(n_1\alpha_1, \ldots, n_j\alpha_j)$. This completes the proof for the constraint $f^{(1)}(\tilde{\mathcal{L}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})_{\leqslant 0} = 0$. The other three constraints can be proved in a similar way. □

8. Concluding remarks

In conclusion, we have investigated the toroidal model of higher-dimensional dispersionless integrable hierarchies proposed by Takasaki. It turns out that such extension admits two classes of dispersionless Lax hierarchies. One is the TdKP hierarchy and the other the TdmKP. For the TdKP, after introducing the dressing approach, the associated Orlov operator can easily be defined. We have established the twistor construction for the TdKP hierarchy and investigated the additional symmetries of the twistor data. Moreover, we have studied Gelfand–Dickey reductions of the TdKP hierarchy and provided some hodograph solutions. The Miura link between the TdKP the TdmKP hierarchies was also studied, which preserve the Lax formulation as well as the twistor construction

Three remarks are in order. Firstly, it would be interesting to discuss the Hamiltonian formulation of the TdKP hierarchy. A possible candidate for conserved quantities would be $H_{n\alpha} = \mathrm{tr}(\mathcal{L}^n\, \mathrm{e}^{\mathrm{i}\alpha\mathcal{U}})/n$ where $\mathrm{tr}\, A = \int \mathrm{res}\, A\, \mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2/2\pi$ which enjoys the properties $\mathrm{tr}\{A, B\} = 0$ and $\mathrm{tr}(\{A, B\}C) = \mathrm{tr}(A\{B, C\})$. It is easy to show that $\partial_{m\beta} H_{n\alpha} = 0$. Besides, we still need to construct the Hamiltonian structure associated with TdKP so that the Lax flows can be written in Hamiltonian flows generated by the above conserved quantities. Secondly, as we see in section 3 and 4 that the dressing operator approach and twistor theoretical construction are independent of that whether θ_1 and θ_2 are compactified coordinates or not. That means the dressing operator approach can be applied to the planar model [33] as well, in which the angle variables (θ_1, θ_2) are replaced by planar coordinates (y, z). In the toroidal model, although we have constructed some solutions for TdKP, yet it is not clear how to find twistor data systematically so that the corresponding hodograph solutions are double periodic. Thirdly, having constructed the Miura transformation between the TdKP and TdmKP hierarchies, it would be interesting to see some multi-dimensional extensions of the modified systems in the explicit form. We hope to work out these issues in the future.

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